Perf is geometric ~~& representability theorem.~~

Before discussing the general result that allows one to check geometricity we will discuss an example where this can be proved by hand. We will follow the arguments given in (Toën-Vaquié - Moduli objects in dg-categories].

First we need to discuss some homological algebra for perfect R-modules. for R a derived ring.

Def'n: For $M \in \mathrm{Perf}(R)$ or Mod_R. and $a \leq b$ two integers. we say M has Tor-amplitude in $[a,b]$ if for any $N \in \mathrm{Mod}_R^{\heartsuit}$ $(\simeq \underline{\mathrm{Mod}}_{H^0(R)})$ one has.

$$M \underset{R}{\otimes} N \in \mathrm{Mod}_R^{\geq a, \leq b}.$$

We will denote by $(\mathrm{Mod}_R)_{[a,b]}$ the subcategory of R-modules of Tor-ampl. in $[a,b]$. and denote by $\mathrm{Perf}(R)_{[a,b]} := (\mathrm{Mod}_R)_{[a,b]} \cap \mathrm{Perf}(R)$.

We list a collection of facts about $\mathrm{Perf}(R)_{[a,b]}$ which we will need in the discuss. of the stack of perfect complexes.

Prop: (i) For $P \in \mathrm{Perf}(R)_{[a,b]}$; $Q \in \mathrm{Perf}(R)_{[c,d]}$ one has $P \underset{R}{\otimes} Q \in \mathrm{Perf}(R)_{[a+c, b+d]}$

(ii) For $f : P \to Q$ w/ $P, Q \in \mathrm{Perf}(R)_{[a,b]}$, then $\mathrm{Fib}(f) \in \mathrm{Perf}(R)_{[a,b+1]}$.

(iii) $P \in \mathrm{Perf}(R)_{[a,b]} \iff P \underset{R}{\otimes} H^0(R) \in \mathrm{Perf}(H^0(R))_{[a,b]}$.

(iv) For any ring map $R \to R'$, $P \in \mathrm{Perf}(R)_{[a,b]} \Rightarrow P \underset{R}{\otimes} R' \in \mathrm{Perf}(R')_{[a,b]}$.

(v) $\mathrm{Perf}(R) = \bigcup_{a \leq b} \mathrm{Perf}(R)_{[a,b]}$.

(vi) $\mathrm{Perf}(R)_{[a,a]} \simeq \mathrm{Vect}(R)[-a]$.

(vii) For any $P \in \mathrm{Perf}(R)_{[a,b]}$, there exist $E \in \mathrm{Vect}(R)$ and a map.

$$E[-b] \xrightarrow{f} P \to \mathrm{Cofib}(f) \quad \text{s.t.} \quad \mathrm{Cofib}(f) \in \mathrm{Perf}(R)_{[a,b-1]}.$$

Idea of proof: (i - iii) are clear from the def'n.
(iv) follows from $\mathrm{oblv}_{R'\to R}: \mathrm{Mod}_{R'} \to \mathrm{Mod}_R$ being t-exact.

(v) follows from (iii) & the classical result.

(vi) Given $P \in \mathrm{Perf}(R)_{[a,a]}$ one has $P[a]$ is flat & almost perfect,
so [HA.7.2.4.20] $\Rightarrow$ $P[a] \in \mathrm{Vect}(R)$. $(\simeq \mathrm{APerf}(R) \cap \mathrm{Flat}(R))$.

For $P \in \mathrm{Perf}(R)_{[a,b]}$
(vii) Consider $E_0 \in \mathrm{Vect}(H^0(R))$ s.t.

$$E_0[-b] \xrightarrow{\bar{f}} P \underset{R}{\otimes} H^0(R) \to \mathrm{Cofib}(\bar{f}) \quad , w/\ \mathrm{Cofib}(\bar{f}) \in \mathrm{Perf}(H^0(R))_{[a,b-1]}.$$

One can find $E \in \mathrm{Vect}(R)$ s.t. $E_0 = E \underset{R}{\otimes} H^0(R)$, i.e. clear for free $H^0(R)$-mods.
+ use that retracts in Mod_R are retracts in $h\mathrm{Mod}_R$.

Thus one has as a lift $f: E[-b] \to P \underset{R}{\otimes} H^0(R) \to P$ and one can check that $\mathrm{Cofib}(f) \in \mathrm{Perf}(R)_{[a,b-1]}$ as required. ∎

Consider the prestack:

$$\mathcal{M} : \mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}.$$
$$S \mapsto \mathrm{Perf}(S)^{\simeq}.$$

Thm: $\mathcal{M}$ is a (locally) geometric stack, i.e. $\mathcal{M} \simeq \mathcal{M}^{(n)}$ where each $\mathcal{M}^{(n)}$ is n-geometric and laft.

$\mathcal{M}$ is a stack.

(1) we notice that if $R \to R'$ is flat one has: suppose $M \in \mathrm{Mod}_R$ s.t. $M \underset{R}{\otimes} R'$ is perfect. In particular, $\exists\ M^{',\mathrm{dual}} \in \mathrm{Mod}_{R'}$ s.t.

$M'^{,\mathrm{dual}} \underset{R'}{\otimes} (-) \simeq \mathrm{Hom}(M \underset{R}{\otimes} R', -)$. By descent for Mod_R let M^{dual} be

s.t. $M^{\mathrm{dual}} \underset{R}{\otimes} R' \simeq M^{',\mathrm{dual}}$. Then M^{dual} shows M is dualizable, whence

M is perfect. By the fact that $\mathrm{QCoh}(-)^{\simeq}$ is a (flat) stack one has.
$\mathcal{M}$ is a stack.

The proposition implies that $\mathcal{M} \simeq \bigcup_{a\le b} \mathcal{M}_{[a,b]}$, where

~~$\mathcal{M}_{QCoh[a,b]}(S) = \{ \mathcal{F} \in Perf(S)_{[a,b]} \}$~~. $\mathcal{M}_{[a,b]}(S) := Perf(S)_{[a,b]}^{\simeq}$.

Thus, it is enough to prove:

Thm': For each pair of integers $a \le b$, the stack $\mathcal{M}_{[a,b]}$ is $(b-a+1)$-geometric.

In the course of the proof of the theorem we will need to understand the space of sections of the stack associated to a single perfect complex.

let $a\le b$ and $b \ge 0$.

Lemma: Let $\mathcal{F} \in Perf(S)_{[a,b]}$ then the stack:

$$Sect(\mathbb{V}(\mathcal{F})) : Sch^{aff}_{/S} \to Spc$$
$$f: T \to S \mapsto \mathrm{Hom}_{QCoh(S)}(\mathcal{F}, f_*\mathcal{O}_T)$$ is b-geometric, i.e.

$Sch^{aff}_{/S} \to Sch^{aff} \xrightarrow{Sect(\mathbb{V}(\mathcal{F}))} Spc$ is b-geometric.

Pf of Lemma: First we consider $b=0$. Then we can take $Sym(\mathcal{F})$ to get an object in $CAlg(QCoh(S))_{\mathcal{O}_S/}$, i.e. $CAlg_{R/}$ where $S = Spec(R)$.

Thus, one has $\mathrm{Hom}_{QCoh(S)}(\mathcal{F}, f_*\mathcal{O}_T) \simeq$ ~~$\mathrm{Hom}_{CAlg(QCoh(S))_{\mathcal{O}_S/}}(Sym(\mathcal{F}), f_*\mathcal{O}_T)$~~

$$\wr\|$$

$$\mathrm{Hom}_{QCoh(T)}(f^*\mathcal{F}, \mathcal{O}_T). \simeq$$ ~~$\mathrm{Maps}_{Sch^{aff}_{/S}}(T, Spec_S(Sym(\mathcal{F})))$~~

$$\wr\|$$

$$\mathrm{Hom}_{CAlg(QCoh(T))_{\mathcal{O}_T/}}(Sym(f^*\mathcal{F}), \mathcal{O}_T) \simeq \mathrm{Maps}_{Sch^{aff}}(T, Spec(Sym(f^*\mathcal{F}))).$$

So $Sect(\mathbb{V}(\mathcal{F}))(T) \simeq Spec(Sym(f^*\mathcal{F}))(T)$, i.e. $Sect(\mathbb{V}(\mathcal{F}))$ is affine. hence 0-geometric.

Inductive step: suppose that for all $\mathcal{F} \in Perf(S)_{[a,b-1]}$ $Sect(\mathbb{V}(\mathcal{F}))$ is $(b-1)$-geometric

Consider $\mathcal{G} \in Perf(S)_{[a,b]}$, let $\mathcal{E} \in Vect(S)$ and $\mathcal{F} \in Perf(S)_{[a,b-1]}$ s.t.

$$\mathcal{E}[-b] \to \mathcal{G} \to \mathcal{F}$$ is a fiber/cofiber sequence.

$f: T \to S.$

Then $\mathrm{Sect}(\mathbb{V}(G))(T) \simeq \mathrm{Hom}_{QCoh(S)}(G, f_* \mathcal{O}_T)$

$$\wr$$

$$\mathrm{Hom}(\mathrm{Cofib}(\mathcal{F} \to \mathcal{E}[-b+1])^{[1]}, f_*\mathcal{O}_T) \simeq \mathrm{Hom}(\mathrm{Cofib}(\mathcal{F} \to \mathcal{E}[-b])^{+1}, {}^{[-1]} f_* \mathcal{O}_T)$$

~~$\mathrm{Fib}(\mathrm{Hom}(\mathcal{E}[-b], f_*\mathcal{O}_T) \xrightarrow{\wr} \mathrm{Hom}(\mathcal{E}[-1], f_*\mathcal{O}_T))$~~ $\wr$

~~So $\Omega\, \mathrm{Sect}(\mathbb{V}(G)) \simeq$~~ $\mathrm{Fib}(\mathrm{Hom}(\mathcal{E}[-b+1], f_*\mathcal{O}_T) \to \mathrm{Hom}(\mathcal{F}, \overset{f_*}{\mathcal{O}_T}))$

So $\Omega\, \mathrm{Sect}(\mathbb{V}(G)) \simeq \mathrm{Fib}(\mathrm{Sect}(\mathbb{V}(\mathcal{E}[-b+1])) \to \mathrm{Sect}(\mathbb{V}(\mathcal{F}))).$

Now we notice. $\mathrm{Sect}(\mathbb{V}(\mathcal{E}[-b+1])) \simeq B^{b-1} \mathbb{G}_a^r$, where $r = \mathrm{rank}\, \mathcal{E}$.

Exercise.

In particular, $\mathrm{Sect}(\mathbb{V}(\mathcal{E}[-b+1]))$ is $(b-1)$-geometric. and by induction so is $\mathrm{Sect}(\mathbb{V}(\mathcal{F}))$. This implies $\Omega\, \mathrm{Sect}(\mathbb{V}(G))$ is $(b-1)$-geometric.

Since. $\mathrm{Sect}(\mathbb{V}(G)) \simeq |\Omega\, \mathrm{Sect}(\mathbb{V}(G))^{\bullet}/*|$ the result follows from the description last time of geometric realizations of gpd. objects in $(b-1)$-geometric stacks. ~~where one notes that~~ ▣

Let $n := b - a + 1$.

<u>Pf of Thm</u>: First we check (i) $\mathcal{M}_{[a,b]}$ has $(n-1)$-geometric diagonal. It is enough to prove that for any $x: S \to \mathcal{M}_{[a,b]}$ & $y: T \to \mathcal{M}_{[a,b]}$ from affine schemes the pullback:

$$S \times_{\mathcal{M}_{[a,b]}} T \text{ is } (n-1)\text{-geometric.}$$

Let $\mathcal{F} \in \mathrm{Perf}(S)_{[a,b]}$ and $G \in \mathrm{Perf}(T)_{[a,b]}$ correspond to x & y.

Notice: $S \times_{\mathcal{M}_{[a,b]}} T \simeq \{ \alpha \in \mathrm{Hom}_{QCoh(S \times T)}(p_1^* \mathcal{F}, p_2^* G) \mid \alpha \text{ is automorphism} \}.$

$$\begin{tikzcd} S \times T \arrow[r, "p_2"'] \arrow[d, "p_1"'] & T \\ S & \end{tikzcd}$$

. One has a map $j: S \times_{\mathcal{M}_{[a,b]}} T \to \mathrm{Sect}(\mathbb{V}(\mathcal{F} \boxtimes G^{\vee}))$

since $\mathrm{Hom}(p_1^* \mathcal{F}, p_2^* G) \simeq \mathrm{Hom}(p_1^* \mathcal{F} \otimes (p_2^* G)^{\vee}, \mathcal{O}_{S \times T}).$

Since $\mathcal{F} \boxtimes G^{\vee} \in \mathrm{Perf}(S \times T)_{[a-b, b-a]}$ by the lemma one has. is $(n-1)$-geom.

So the result will follow from proving j is representable by an aff. scheme. This follows from noticing that

$$\begin{tikzcd}
S \times_{M[a,b]} T \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sect}(\mathbb{V}(\mathcal{F} \boxtimes \mathcal{G}^{\vee})) \arrow[d] \\
\tau_{\leq 0}(S \times_{M[a,b]} T) \arrow[r, "\tau_{\leq 0} j"'] & \tau_{\leq 0}\, \mathrm{Sect}(\mathbb{V}(\mathcal{F} \boxtimes \mathcal{G}^{\vee}))
\end{tikzcd}$$

where $\tau_{\leq 0} X$ denotes the 0-truncated stack associated to X.

and that $\tau_{\leq 0} j$ is affine representable. ■

Now we check (ii): $M_{[a,b]}$ has a smooth & surjective atlas from a $(n-1)$-geom. stack.

Naturally, we proceed by induction on $n = b - a + 1$. The case $n=1$ we argued last time, i.e. $M_{[a,a]} \simeq \mathrm{Vect}$ which is 1-geometric.

Consider the stack U defined by the following pullback diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], \mathrm{Perf})^{\simeq} \arrow[d, "ev_0 \times ev_1"] \\
M_{[a,b-1]} \times \mathrm{Vect} \arrow[r] & \mathrm{Perf}^{\simeq} \times \mathrm{Perf}^{\simeq} \\
(M, N) \arrow[r, mapsto] & (M, N[-b+1]) .
\end{tikzcd}$$

Concretely, one has $U(S) = \{ M \in \mathrm{Perf}(R)_{[a,b-1]}, N \in \mathrm{Vect}(R), \varphi : M \to N[-b+1] \}$.

Notice
$$p : U \to M_{[a,b]}.$$
$$(M, N, \varphi) \mapsto \mathrm{Fib}(\varphi).$$
which has Tor amplitude in $[a,b]$ by the first Prop. today

We need to prove: ① U is $(n-1)$-geometric.

② p is smooth & surjective.

For ① we consider. $h : U \to M_{[a,b-1]} \times \mathrm{Vect}$, since by induction $M_{[a,b-1]} \times \mathrm{Vect}$ is $(n-1)$-geometric.

We just need to check that $\bar{n}$ is $(n-1)$-geometric.

Given a fixed pair $(M, N) \in \mathcal{M}_{[a,b-1]}(S) \times \mathrm{Vect}(S)$ we have:

$$\bar{n}^{-1}(S) \simeq \mathrm{Hom}_{QCoh(S)}(M, N[-b+1]) \simeq \mathrm{Hom}_{\mathrm{Mod}_R}(M \otimes N^\vee[b-1], R)$$

Since $M \otimes N^\vee[b-1] \in \mathcal{M}_{[a-b+1,0]}$, one has $\bar{n}^{-1}(S)$ is affine representable.

For ② it is ~~easy to see~~ clear that p is surjective, i.e. from Prop. (viii).

Recall that p is smooth if $\forall\, x: S \to \mathcal{M}_{[a,b]}$, $p^{-1}(S) \to S$ is smooth as a $(n-1)$-geometric stack.

Consider the pull back diagram:

$$\begin{tikzcd}
V_r := S \times_{\mathrm{Vect}\times S} p^{-1}(S) \arrow[rr] \arrow[ddd, "q_r"'] & & U \times_{\mathcal{M}_{[a,b]}} S = p^{-1}(S) \arrow[d] \\
& & U \times S \arrow[d] \\
& & \mathcal{M}_{[a,b-1]} \times \mathrm{Vect} \times S \arrow[d] \\
\coprod_{r \geq 0} S \arrow[r] \arrow[rr, bend right, "u_r"'] & \coprod_{r \geq 0} \mathrm{Vect}_r \times S \arrow[r, "\simeq"] & \mathrm{Vect} \times S
\end{tikzcd}$$

Since $\coprod_r S \to \mathrm{Vect} \times S$ is a smooth cover it is enough to check that for each $r \geq 0$ $\overset{V_r}{\overset{\|}{S \times_{\mathrm{Vect}\times S} p^{-1}(S)}} \xrightarrow[q_r]{} S$ is smooth.

Notice that for $f: T \to S$ one has: $V_r \times_S T = \mathrm{Hom}_{QCoh(T)}(\mathcal{O}_T^r[-b+1], f^* P[1])$

where $P \in \mathrm{Perf}(S)_{[a,b]}$ corresponds to x fixed above.

and $\mathrm{Hom}_{QCoh(T)}(\mathcal{O}_T^r[-b+1], f^* P[1]) \simeq \mathrm{Hom}_{QCoh(S)}(\mathcal{O}_S^r \otimes P^\vee[-b], \mathcal{O}_T)$.

Given any sheaf $\mathcal{F} \in \mathrm{QCoh}(T)^{\leq 0}$ one clearly has:

$$\mathrm{Maps}_{\otimes T/-}(T_{\mathcal{F}}, V) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathcal{O}_S^r \otimes P^\vee[-b], \mathcal{F}) \wr \mathrm{Hom}_{\mathrm{QCoh}(T)}(\mathcal{O}_T^r \otimes f^* P^\vee[-b], \mathcal{F}).$$

Thus, $T^*(V/S) = \mathcal{O}_T^r \otimes f^* P^\vee[-b] \in \mathrm{Perf}(T)_{[0,b-a]}$.

Now we invoke the following fact that we haven't proved but is believable from the discussions of the last two talks.

A map $f: \mathcal{X} \to \mathcal{Y}$ between n-geometric stacks lafp is smooth iff $\forall S \xrightarrow{x} \mathcal{X}$ affine. & $\mathcal{H}$ ~~$\mathcal{F}$~~ $\in \mathrm{QCoh}(S)^\heartsuit$

$$\mathrm{Hom}_{h\mathrm{QCoh}(S)}(T_x^*(\mathcal{X}/\mathcal{Y}), \mathcal{H}[1]) \simeq 0.$$

Since $\mathrm{Hom}_{h\mathrm{QCoh}(S)}(T_x^*(\mathcal{X}/\mathcal{Y}), \mathcal{H}[1]) \simeq H^0((T_x^*\mathcal{O}(\mathcal{X}/\mathcal{Y}))^\vee[1] \otimes \mathcal{H})$.

and $T_x^*(\mathcal{X}/\mathcal{Y})^\vee[1] \in \mathrm{Perf}(S)_{[a-b-1,-1]}$.

$= 0$.

So f is smooth. This finishes the proof. ∎